

FIRST ORDER OPTIMUM CALCULI

ANDRZEJ BOROWIEC

Institute of Theoretical Physics, University of Wrocław

Plac Maxa Borna 9, 50204 Wrocław, Poland

E-mail: borowiec@ift.uni.wroc.pl

VLADISLAV K. KHARCHENKO

Institute of Mathematics, Novosibirsk, Russia

E-mail: kharchen@math.nsk.su

Abstract: A new notion of an optimum first order calculi was introduced in [Borowiec, Kharchenko and Oziewicz, 1993]. A module of vector fields for a coordinate differential is defined. Some examples of optimal algebras for homogeneous bimodule commutations are presented. Classification theorem for homogeneous calculi with commutative optimal algebras in two variables is proved.

1. Introduction. Quantum spaces are identified with noncommutative algebras. Differential calculi on quantum spaces have been elaborated by Pusz and Woronowicz [6], by Pusz [5] and by Wess and Zumino [7]. A bicovariant differential calculus on quantum groups was presented by Woronowicz [8]. Woronowicz found that a construction of a first order calculus, a bimodule of one forms, is not functorial. There are many nonequivalent calculi for a given associative algebra. This yields a problem of classification of first order calculi.

In our previous paper [1] following the developments by Pusz, Woronowicz [6] and by Wess, Zumino [7], we have proposed a general algebraic formalism for first order calculi on an arbitrary associative algebra with a given presentation. A basic idea was that a noncommutative differential calculus is best handled by means of commutation relations between generators of an algebra and its differentials. We are assuming that a bimodule of one forms is a free right module. This allows to define partial derivatives (vector fields). Corresponding calculi and a differential are said to be a *coordinate calculi* and a *coordinate differential*. Any coordinate differential defines a commutation rule $vd x^i = dx^k \cdot A(v)_k^i$, where $A : v \mapsto A(v)_k^i$ is an algebra homomorphism $A : R \rightarrow R_{n \times n}$. It is easy to see that for any homomorphism $R \rightarrow R_{n \times n}$ there exists not more than one coordinate differential. We have considered the existence problem of such a differential. We showed that for a given commutation rule $vd x^i = dx^k \cdot A(v)_k^i$ a free algebra generated by the variables x^1, \dots, x^n has a related coordinate differential. We are defining an *optimal* algebra with respect to a fixed commutation rule. In the homogeneous case this algebra is characterized as the unique algebra which has no nonzero A -invariant subspaces with zero differentials.

In the section II we shall recall the general formalism from ref. [1]. In the section III we shall consider a number of examples of optimal algebras for given

1991 Mathematics Subject Classification: 16D20; 16W25; 16U80.

This work has been supported by grants from State Research Committee KBN No 2 P302 02307 (A. B.) and Russian Foundation for Fundamental Research No 93-01-16171 (V. K. K.).

commutation rules. The last section is devoted to classification of commutation rules in two variables which determine commutative optimal algebra.

2. Coordinate differentials and optimal algebras. In this section we mainly review our results from [1]. The proofs of all theorems in this section are in ref. [1].

Let F be a field. Throughout this paper, an algebra means an unital associative F -algebra generated by a F -space V with a basis x^1, \dots, x^n . A presentation of an algebra R is an epimorphism $\pi : F \langle \hat{x}^1, \dots, \hat{x}^n \rangle \rightarrow R$, where $F \langle \hat{x}^1, \dots, \hat{x}^n \rangle$ is the free associative unital algebra generated by the variables $\hat{x}^1, \dots, \hat{x}^n$. Let $I_R = \ker \pi$, then $R \cong F \langle \hat{x}^1, \dots, \hat{x}^n \rangle / I_R$, I_R is an (twosided) ideal of relations in R and $\pi(\hat{x}^i) = x^i$ for $i = 1, \dots, n$.

A differential from an algebra R to a bimodule M is a linear map satisfying the Leibniz rule

$$d(uv) = (du)v + u dv.$$

DEFINITION 2.1 A differential $d : R \rightarrow M$ is said to be *coordinate* if bimodule M is a free right R -module freely generated by dx^1, \dots, dx^n .

NOTE: Coordinate calculi were studied in ref. [1] under the name *free calculi*, due to the freeness condition for the bimodule M of one forms. More general calculi, *calculi with partial derivatives* and their extensions to higher order (quantum de Rham complexes) have been recently considered by the same authors in [4].

If d is a coordinate differential, then a linear maps $D_k : R \rightarrow R$, partial derivatives, are uniquely defined by the formula,

$$d v = dx^k \cdot D_k(v) \quad (2.1)$$

Then

$$D_k(x^i) = \delta_k^i, \quad (2.2)$$

where δ_k^i is the Kronecker delta.

PROPOSITION 2.2 A linear map $A_d : R \rightarrow R_{n \times n}$ from an algebra R into the algebra of n by n matrices over R given by the formula

$$A_d(v)_k^i \equiv D_k(vx^i) - D_k(v)x^i \quad (2.3)$$

is an algebra homomorphism i.e.

$$A_d(uv)_k^i = A_d(u)_k^l A_d(v)_l^i \quad (2.4)$$

The partial derivatives D_k and a homomorphism A_d are connected by the relation

$$D_k(uv) = D_k(u)v + A_d(u)_k^i D_i(v) \quad (2.5)$$

The inverse statement also holds.

PROPOSITION 2.3 *Let R be an algebra generated by x^1, \dots, x^n and $A : R \rightarrow R_{n \times n}$ be an algebra homomorphism. If $D_k : R \rightarrow R$, $k = 1, \dots, n$ are linear maps such that*

$$D_k(x^i) = \delta_k^i \quad (2.6)$$

$$D_k(uv) = D_k(u)v + A(u)_k^i D_i(v), \quad (2.7)$$

then the map $\Delta : v \mapsto dx^k \cdot D_k(v)$ is a coordinate differential, where $M_\Delta = \sum dx^i \cdot R$ is a free right module with the left module structure defined by commutation rule $vd x^i = dx^k A(v)_k^i$ i.e. $A_\Delta = A$.

Let M^* be a free left R -module freely generated by the partial derivatives D_i , i.e. $Y \in M^*$ iff $Y = Y^i D_i$ with $Y^i \in R$. Define a right R -module structure on M^* by the transpose commutation rule

$$Y.v \equiv Y^i (D_i.v) = Y^i A(v)_i^k D_k$$

Thus we have defined a bimodule M^* of vector fields as a dual to a bimodule M of differential forms together with a pairing $\langle Y, \omega \rangle \equiv Y^i \omega_i \in R$, where $\omega = dx^i \cdot \omega_i \in M$. Then $\langle Y.f, \omega \rangle = \langle Y, f\omega \rangle$. A vector field $Y \in M^*$ can be characterized as a linear map (endomorphism) $Y : R \rightarrow R$ satisfying the twisted Leibniz's rule vis.

$$Y(uv) = Y(u)v + (Y.u)(v) \quad (2.8)$$

where $Y(u) = Y^i D_i(u)$ and $Y.u$ is a previously defined multiplication from the right. It generalizes the formulae (2.5) to arbitrary vector field Y .

Both definitions differential forms and vector fields essentially depend on the generating space $V = \text{lin}(x^1, \dots, x^n)$ in the following sense. Let $z^k = \alpha_i^k x^i$ be another base in F -space V with a matrix $(\alpha_i^k) \in GL(n, F)$. Then the basis of a bimodule M , a differentials dx^i and basic vector fields $D_i = \partial/\partial x^i$ undergo correspondingly covariant and contravariant transformation law, i.e.

$$dz^k = \alpha_i^k dx^i, \quad \partial/\partial z^k = \beta_k^i \partial/\partial x^i$$

where $\beta_l^k \alpha_i^l = \delta_i^k$.

A question concerning Proposition 2.3 arises here. If a homomorphism A is given, then formula (2.7) allows one to calculate partial derivatives of a product in terms of its factors. That fact and formula (2.6) show that for a given A there exists not more then one D satisfying formulas (2.6) and (2.7). It is not clear yet whether or not there exists at least one D of such a type. Thus, our first task is to describe these homomorphisms of A for which there exist coordinate differentials with $A_d = A$.

THEOREM 2.4 *Let $R = F \langle x^1, \dots, x^n \rangle$ be a free algebra generated by x^1, \dots, x^n and A^1, \dots, A^n be any set of $n \times n$ matrices over R . There exists the unique coordinate differential d such that $A_d(x^i) = A^i$.*

Let R be an F -algebra defined by a set of generators x^1, \dots, x^n and a set of relations $f_m(x^1, \dots, x^n) = 0$, $m \in \mathcal{M}$ i.e. $R = \hat{R} / \hat{I}$, where $\hat{R} = F \langle \hat{x}^1, \dots, \hat{x}^n \rangle$ is a free algebra and \hat{I} is its ideal generated by elements $f_m(\hat{x}^1, \dots, \hat{x}^n)$, $m \in \mathcal{M}$.

Let π be an algebra projection $\hat{R} \rightarrow R$ such that $\pi(\hat{x}^i) = x^i$, $\pi(1) = 1$. Since $R_{n \times n} = R \otimes F_{n \times n}$, π defines an epimorphism $\hat{\pi} : \hat{R}_{n \times n} \rightarrow R_{n \times n}$ by the formula $\hat{\pi} = \pi \otimes id$.

If $A : R \rightarrow R_{n \times n}$ is an algebra map, then we have the following diagram of algebra homomorphisms

$$\begin{array}{ccccc} \hat{R} & \xrightarrow{\hat{A}} & \hat{R} \otimes F_{n \times n} & = & \hat{R}_{n \times n} \\ \pi \downarrow & & \downarrow \pi \otimes id & = & \downarrow \hat{\pi} \\ R & \xrightarrow{A} & R \otimes F_{n \times n} & = & R_{n \times n} \end{array} \quad (2.9)$$

Let choose for each generator x^i an arbitrary element $\hat{A}^i \in \hat{R}$ such that $\hat{\pi}(\hat{A}^i) = A^i$. Then the map $\hat{x}^i \mapsto \hat{A}^i$ can be extended to an algebra homomorphism $\hat{A} : \hat{R} \rightarrow \hat{R}_{n \times n}$ (recall that $\hat{x}^1, \dots, \hat{x}^n$ are free variables). This homomorphism completes a diagram (2.9) to a commutative diagram. For each relation $f_m(\hat{x}^1, \dots, \hat{x}^n)$ we have:

$$\hat{\pi}(\hat{A}(f_m)) = A(\pi(f_m)) = 0. \quad (2.10)$$

Furthermore,

$$ker \hat{\pi} = ker(\pi \otimes id) = ker \pi \otimes F_{n \times n} = \hat{I}_{n \times n},$$

and finally,

$$\hat{A}(f_m) \in ker \hat{\pi} = \hat{I}_{n \times n}.$$

Theorem 2.4 claims that for a homomorphism $\hat{A} : \hat{R} \rightarrow \hat{R}_{n \times n}$ there exists a unique coordinate differential \hat{d} of a free algebra \hat{R} .

DEFINITION 2.5 The differential \hat{d} is called a *cover* differential with respect to the homomorphism $A : R \rightarrow R_{n \times n}$.

We have proved:

THEOREM 2.6 *For any homomorphism $A : R \rightarrow R_{n \times n}$ there exists a cover differential \hat{d} of a free algebra \hat{R} .*

PROPOSITION 2.7 *Let an algebra R be generated by x^1, \dots, x^n subject to the relations $\{f_m = 0, m \in \mathcal{M}\}$. Let \hat{D}_k be a partial derivatives of the cover differential \hat{d} ,*

and \hat{I} be an ideal generated by $\{f_m, m \in \mathcal{M}\}$. Then an algebra R possesses a coordinate differential with respect to a homomorphism $A : R \rightarrow R_{n \times n}$ if and only if

$$\hat{D}_k(f_m(\hat{x}^1, \dots, \hat{x}^n)) \in \hat{I}$$

COROLLARY 2.8 *Let an algebra R be generated by x^1, \dots, x^n subject to the set of homogeneous relations $\{f_m = 0, m \in \mathcal{M}\}$ of the same degree. If $A : R \rightarrow R_{n \times n}$ acts linearly on generators $A(x^j)_k^i = \alpha_{kl}^{ij} x^l$, then for a pair R, A there exists a coordinate differential if and only if for all $m \in \mathcal{M}$*

$$\hat{d}f_m = 0 \quad . \quad (2.11)$$

DEFINITION 2.9 An ideal $I \neq \hat{R}$ of a free algebra $\hat{R} = F \langle \hat{x}^1, \dots, \hat{x}^n \rangle$ is said to be *consistent* with a homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$ if a factor algebra \hat{R}/I has a coordinate differential satisfying the commutation rules

$$x^j dx^i = dx^k \cdot A(x^j)_k^i \quad .$$

If an ideal I is A -consistent, then Proposition 2.2 defines a homomorphism $A : r \mapsto A_k^i(r)$ from the factor algebra into the matrix algebra over it. Thanks to Proposition 2.7 it follows that I is A -consistent iff $A_k^i(I) \subseteq I$, and if $D_k(I) \subseteq I$ for any of partial derivatives D_k defined by a differential d corresponding to A (see Theorem 2.4).

These two condition generalize Wess and Zumino's quadratic and a linear consistency conditions for quadratic algebras [7].

We have proved that free algebra \hat{R} admits a coordinate differential for arbitrary commutation rules. In order to define a homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$ it is enough to set its value on generators vis.

$$A_k^i(x^m) = \alpha_k^{mi} + \alpha_{kj_1}^{mi} x^{j_1} + \alpha_{kj_1 j_2}^{mi} x^{j_1} x^{j_2} + \dots$$

where $\{\alpha_{kj_1, \dots, j_r}^{mi}\}$ are arbitrary tensor coefficients. If a homomorphism A preserves a degree, then it must act linearly on generators $A_k^j(\hat{x}^i) = \alpha_{kl}^{ij} \hat{x}^l$. Therefore, a homomorphism A is defined by a 2-covariant and 2-contravariant tensor $A = \alpha_{kl}^{ij}$. This is a homogeneous case. The case $A_k^j(x^i) = \alpha_k^{ij} + \alpha_{kl}^{ij} x^l$ has been considered by Dimakis and Müller–Hoissen [2]. In this case the partial derivatives do not increase the degree.

For any homomorphism A there exists the largest A -consistent ideal $I(A)$ contained in the ideal \bar{R} of polynomials with zero constant terms ($\bar{R} + F \cdot 1 = \hat{R}$) – the sum of all consistent ideals of such a type.

NOTE: An ideal $I(A)$ need not to be the only maximal A -consistent ideal in \hat{R} .

THE MAIN DEFINITION. The factor algebra $R_A = \hat{R}/I(A)$ is said to be an *optimal algebra* for the commutation rule $x^i dx^j = dx^k \cdot A(x^i)_k^j$. A pair: the optimal algebra R_A , and the differential d corresponding to the commutation rule A will be called an *optimum calculus*.

We will describe an ideal $I(A)$ in the homogeneous case.

THEOREM 2.10 For each 2-covariant 2-contravariant tensor $A = \alpha_{kl}^{ij}$ the ideal $I(A)$ can be constructed by induction as the homogeneous space $I(A) = I_1(A) + I_2(A) + I_3(A) + \dots$ in the following way:

- (i) $I_1(A) = 0$
- (ii) Assume that $I_{s-1}(A)$ has been defined and U_s be a space of all polynomials m of degree s such that $D_k(m) \in I_{s-1}(A)$ for all k , $1 \leq k \leq n$. Then $I_s(A)$ is the largest A -invariant subspace of U_s .

The ideal $I(A)$ is a maximal A -consistent ideal in \hat{R} .

COROLLARY 2.11 The Theorem 2.10 shows in particular that if a homogeneous element is such that all elements of the invariant subspace generated by it have all partial derivatives equal to zero, then that element vanishes in the optimal algebra.

3. Examples of optimum calculi. In this section we shall consider a number of commutation rules A and the corresponding optimal algebras R_A .

EXAMPLE 3.1. Consider a diagonal commutation rule, $x^j dx^i = dx^i \cdot q^{ij} x^j$, with $q^{ij} q^{ji} = 1, i \neq j$. If no one of the coefficients q^{ii} is a root of a polynomial of the type $\lambda^{[m]} \doteq \lambda^{m-1} + \lambda^{m-2} + \dots + 1$, then the optimal algebra is $R_A = F \langle x^1, \dots, x^n \rangle / \{q^{ij} x^i x^j = x^j x^i, i < j\}$.

If $(q^{ii})^{[m_i]} = 0, 1 \leq i \leq s$ with minimal m_i then

$R_A = F \langle x^1, \dots, x^n \rangle / \{q^{ij} x^i x^j = x^j x^i, i < j, (x^i)^{m_i} = 0, 1 \leq i \leq s\}$.

Proof. First of all we have to note that the elements $q^{lj} x^l x^j - x^j x^l$ are zero in the optimal algebra. By Theorem 2.10 it is sufficient to show that all partial derivatives (for cover differential) of elements of the invariant space generated by this forms are equal to zero. In our example $A(x^j)_k^i = \delta_k^i q^{ij} x^j$ and therefore

$$\begin{aligned} A(q^{lj} x^l x^j - x^j x^l)_k^i &= q^{lj} A(x^l)_s^i A(x^j)_k^s - A(x^j)_s^i A(x^l)_k^s = \\ &= q^{lj} \delta_s^i q^{il} x^l \cdot \delta_k^s q^{sj} x^j - \delta_s^i q^{ij} x^j \cdot \delta_k^s q^{sl} x^l = \\ &= \delta_k^i q^{kl} q^{kj} [q^{lj} x^l x^j - x^j x^l] \end{aligned} \tag{3.1}$$

So it is enough to check that the partial derivatives of the relations are zero:

$$\begin{aligned} D_k(q^{lj}x^lx^j - x^jx^l) &= q^{lj}[D_k(x^l)x^j + A(x^l)_k^i D_i(x^j)] - \\ &\quad - [D_k(x^j)x^l - A(x^j)_k^i D_i(x^l)] = 0 \end{aligned}$$

Formula (3.1) and Corollary 2.7 show that the factor algebra $S = F \langle x^1, \dots, x^n \rangle / \{q^{ij}x^i x^j = x^j x^i, \ i < j\}$ has coordinate differential with our commutation rules. For this algebra to be optimal by Theorem 2.10 it is enough to see that any homogeneous element of positive degree with zero all the partial derivatives is equal to zero in this algebra.

We have

$$\begin{aligned} D_k[(x^j)^m] &= \delta_k^j (x^j)^{m-1} + A(x^j)_k^i D_i[(x^j)^{m-1}] = \\ &= \delta_k^j (x^j)^{m-1} + \delta_k^i q^{ij} x^j D_i[(x^j)^{m-1}] \end{aligned}$$

and by an easy induction

$$D_k[(x^j)^m] = \delta_k^j [1 + q^{jj} + (q^{jj})^2 + \dots + (q^{jj})^{m-1}] (x^j)^{m-1}. \quad (3.2)$$

An arbitrary element of the algebra S has a unique presentation of the form

$$f = \sum \alpha_i (x^1)^{i_1} (x^2)^{i_2} \dots (x^n)^{i_n},$$

where, $i \equiv i_1 i_2 \dots i_n$ is a multi index. Thus by formula (3.2)

$$\begin{aligned} D_k f &= \sum \alpha_i (x^1)^{i_1} (x^2)^{i_2} \dots D_k[(x^k)^{i_k}] \dots (x^n)^{i_n} = \\ &= \sum (q^{kk})^{[i_k-1]} \alpha_i (x^1)^{i_1} (x^2)^{i_2} \dots (x^k)^{i_k-1} \dots (x^n)^{i_n}. \end{aligned} \quad (3.3)$$

If no one of the elements $(q^{kk})^{[m]}$ is zero, then the right hand side components of the formula (3.3) are equal to zero for all k if and only if $\alpha_i = 0$. So $f = 0$ and S is the optimal algebra.

If $(q^{ii})^{[m_i]} = 0$, $1 \leq i \leq s$, then by (3.2) we have $D_k[(x^i)^{m_i}] = 0$ and also $D_k[A\{(x^j)^{m_j}\}] = 0$ because of $A(x^j)_k^i = \delta_k^i q^{ij} x^j$ implies that $A\{(x^j)^{m_j}\}$ has the form $\Lambda \cdot (x^j)^{m_j}$, where Λ is a matrix with coefficients from the base field. Therefore $(x^j)^{m_j} = 0$ in the optimal algebra.

Each element of the algebra

$$\bar{S} = F \langle x^1, \dots, x^n \rangle / \{q^{ij}x^i x^j = x^j x^i, \ i < j, \ (x^i)^{m_i} = 0, \ 1 \leq i \leq s\}$$

has unique presentation of the form (3.1), where $i_1 < m_1, i_2 < m_2, \dots, i_s < m_s$. Now formulae (3.2) and (3.3), which are still valid in \bar{S} , imply that if all the partial derivatives of an element are zero in \bar{S} then this element is zero in \bar{S} . Q.E.D.

EXAMPLE 3.2. Let $A = 0$ i.e. $x^i dx^j = 0$. Then \hat{d} is a homomorphism of right modules and the optimal algebra is free $R_A = \hat{R}$.

Proof: Indeed, if $u = x^1 u_1 + \dots + x^n u_n$ then $D_k(u) = u_k$ and any element with zero partial derivatives is zero. Thus by Theorem 2.10 the ideal \hat{I}_A contains no nonzero elements.
Q.E.D.

Note that the calculus with this commutation rule in fact was defined in Fox's paper [3]. This calculus is essential tool in series of Fox's papers on group theory. It was defined for the free group algebra by the following formula for derivations:

$$D(uv) = D(u)v + u^\circ D(v)$$

where, u° is the sum of all coefficient of the element u from group algebra of free group G freely generated by the elements x^1, \dots, x^n .

If partial derivatives D_k for a differential d obey this conditions then the commutation rule has the form $x^i dx^j = dx^j$. If we change the variables $X^i = x^i - 1$ than we will obtain the calculus with zero homomorphism A .

EXAMPLE 3.3. Let $x^i dx^j = -dx^i \cdot x^j$. Then the optimal algebra is the smallest possible algebra generated by the space V i.e. $R_A = F \langle x^1, \dots, x^n \rangle / \{x^i x^j = 0\} = F + V$.

Proof: In this case $\hat{d}(x^i x^j) = \hat{d}x^i \cdot x^j + x^i \hat{d}x^j = 0$ for cover differential. Evidently the space of all quadratic forms is A -invariant. By Theorem 2.10 in the optimal algebra $x^i x^j = 0$.
Q.E.D.

EXAMPLE 3.4. Let $x^1 dx^1 = dx^1 \cdot (\alpha_2 x^2 + \dots + \alpha_n x^n)$ and $x^i dx^j = -dx^i \cdot x^j$ if $i \neq 1$ or $j \neq 1$. Then the optimal algebra is almost isomorphic to the ring of polynomials in one variable. More precisely, $R_A = F \langle x^1, \dots, x^n \rangle / \{x^i x^j = 0, \text{ unless } i = j = 1\}$.

Proof. In this case $A_k^1(x^1) = \delta_k^1(\alpha_2 x^2 + \dots + \alpha_n x^n)$ and $A_k^j(x^i) = -\delta_k^i x^j$ if $i \neq 1$ or $j \neq 1$. Let I be an ideal generated by all products $x^i x^j$ but $x^1 x^1$ and let S be a factor algebra \hat{R}/I . We have

$$\hat{D}_k(x^i x^j) = \delta_k^i x^j + A_k^s(x^i) \delta_s^j = \delta_k^i x^j + A_k^j(x^i) = 0$$

if either $i \neq 1$ or $j \neq 1$. We also have

$$A_s^m(x^i x^j) = A_k^m(x^i) A_s^k(x^j)$$

Both left and right factors contain an addendum of the form αx^1 only if both $m = k = 1$ and $i = k, j = s$ are valid and both $m = i = 1$ and $j = k = 1$ are false, which is impossible. Thus the product belong to the ideal I . By the Proposition

2.7 I is a consistent ideal.

Any element of the factor algebra S has unique presentation of the form

$$f = \gamma_k x^k + \beta_2(x^1)^2 + \dots + \beta_N(x^1)^N$$

For $k \neq 1$, we have

$$D_k[(x^1)^N] = A_k^s(x^1)D_s[(x^1)^{N-1}] = -x^1 D_1[(x^1)^{N-1}]$$

and

$$\begin{aligned} D_1[(x^1)^N] &= (x^1)^{N-1} + A_1^k(x^1)D_k[(x^1)^{N-1}] = (x^1)^{N-1} + w \cdot D_1[(x^1)^{N-1}] - \\ &\quad - \sum_{k \geq 2} x^k \cdot D_k[(x^1)^{N-1}] = \begin{cases} (x^1)^{N-1} & \text{if } N \geq 3, \\ x^1 + w & \text{if } N = 2, \end{cases} \end{aligned}$$

where $w = \alpha_2 x^2 + \dots + \alpha_n x^n$. Now if $df = 0$ in S then $\gamma_k = D_k(f) = 0$ for $k \geq 2$ and

$$D_1 f = \gamma_1 + \beta_2 x^1 + \beta_2 w + \beta_3 (x^1)^2 + \dots + \beta_N (x^1)^{N-1} = 0$$

Therefore $\beta_1 = \dots = \beta_N = 0$ and $R_A = S$.

Q.E.D.

EXAMPLE 3.5. Let $\mu, \lambda \in F$. If $n = 2$ and $x^1 dx^1 = dx^1 \cdot \mu x^2$, $x^1 dx^2 = -dx^1 \cdot x^2$, $x^2 dx^1 = -dx^2 \cdot x^1$, $x^2 dx^2 = dx^2 \cdot \lambda x^1$, then the optimal algebra is isomorphic to the direct sum of two copies of the polynomial algebra $R_A = F \langle x^1, x^2 \rangle / \{x^1 x^2 = x^2 x^1 = 0\}$.

Proof. Let I be the ideal generated by $x^1 x^2$ and $x^2 x^1$. We have $\hat{d}(x^1 x^2) = \hat{d}(x^2 x^1) = 0$. By definition of this commutation rule we have

$$\begin{aligned} A(x^1) &= \begin{pmatrix} \mu x^2 & -x^2 \\ 0 & 0 \end{pmatrix}; & A(x^2) &= \begin{pmatrix} 0 & 0 \\ -x^1 & \lambda x^1 \end{pmatrix}; \\ A(x^1 x^2) &= \begin{pmatrix} x^2 x^1 & -\lambda x^2 x^1 \\ 0 & 0 \end{pmatrix} \equiv 0 \pmod{I}; \\ A(x^2 x^1) &= \begin{pmatrix} 0 & 0 \\ -\mu x^1 x^2 & x^1 x^2 \end{pmatrix} \equiv 0 \pmod{I}. \end{aligned}$$

Therefore the ideal I is consistent.

In the algebra $S \equiv F \langle x^1, x^2 \rangle / I$ any element has a unique presentation of the form

$$f = \alpha_1 x^1 + \alpha_2 (x^1)^2 + \dots + \alpha_n (x^1)^n + \beta_1 x^2 + \beta_2 (x^2)^2 + \dots + \beta_m (x^2)^m$$

We have $D_2[(x^1)^n] = A_2^k(x^1) \cdot D_k[(x^1)^{n-1}] = 0$ and therefore

$$D_1[(x^1)^n] = (x^1)^{n-1} + A_1^k(x^1) \cdot D_k[(x^1)^{n-1}] =$$

$$= (x^1)^{n-1} + \mu x^2 \cdot D_1[(x^1)^{n-1}] - x^2 \cdot D_2[(x^1)^{n-1}] = (x^1)^{n-1} + (\mu x^2)^{n-1}$$

By this formula

$$D_1 f = \alpha_1 + \alpha_2 x^1 + \dots + \alpha_n (x^1)^{n-1} + \alpha_2 \mu x^2 + \alpha_3 (\mu x^2)^2 + \dots + \alpha_n (\mu x^2)^{n-1}$$

Therefore $D_1 f = 0$ only if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$. Analogously, $D_2 f = 0$ only if $\beta_1 = \beta_2 = \dots = \beta_m = 0$. By Theorem 2.10 in this case the optimal algebra is S . Q.E.D.

4. Homogeneous commutation rules in two variables with commutative optimal algebra. In this section we will describe all homogeneous commutation rules in two variables with a commutative optimal algebra. In this case the ideal $I(A)$ is homogeneous and is defined by Theorem 2.10. Commutativity of the optimal algebra is equivalent to $x^1 x^2 - x^2 x^1 \in I_2$, where I_2 is the second homogeneous component of $I(A)$. We will call the commutation rule (and the corresponding optimal algebra) *regular* if the space I_2 is one dimensional, i.e. if it is generated by commutator. In the opposite case we will call the commutation rule and the optimal algebra *irregular*. For instance the optimum calculi in the Examples 3–5 are irregular. Evidently if the optimal algebra is isomorphic to the algebra of polynomials in two variables then the commutation rule is regular (but not vice versa).

THEOREM 4.1 *Let $u, v, w, v_1 \in V = \text{lin}(x^1, x^2)$ and $\lambda, \mu \in F$. A homogeneous commutation rule with regular commutative optimal algebra belongs (up to renaming of variables $x^1 \leftrightarrow x^2$) to one of the following four classes:*

(I)

$$\begin{aligned} x^1 dx^1 &= dx^1 \cdot u + dx^2 \cdot v, \\ x^1 dx^2 &= dx^1 \cdot w + dx^2 \cdot (\lambda v + x^1), \\ x^2 dx^1 &= dx^1 \cdot (w + x^2) + dx^2 \cdot (\lambda v), \\ x^2 dx^2 &= dx^1 \cdot (\lambda w) + dx^2 \cdot (\lambda^2 v - \lambda u + w + \lambda x^1 + x^2); \end{aligned}$$

(II)

$$\begin{aligned} x^1 dx^1 &= dx^1 \cdot (x^1 + \mu v + v_1) + dx^2 \cdot v, \\ x^1 dx^2 &= dx^1 \cdot \lambda v + dx^2 \cdot (v_1 + x^1), \\ x^2 dx^1 &= dx^1 \cdot (\lambda v + x^2) + dx^2 \cdot v_1, \\ x^2 dx^2 &= dx^1 \cdot \lambda v_1 + dx^2 \cdot (\lambda v - \mu v_1 + x^2); \end{aligned}$$

(III)

$$\begin{aligned} x^1 dx^1 &= dx^1 \cdot u, \\ x^1 dx^2 &= dx^2 \cdot x^1, \\ x^2 dx^1 &= dx^1 \cdot x^2, \\ x^2 dx^2 &= dx^2 \cdot v; \end{aligned}$$

(IV)

$$\begin{aligned}
x^1 dx^1 &= dx^1 \cdot u, \\
x^1 dx^2 &= dx^2 \cdot u, \\
x^2 dx^1 &= dx^1 \cdot x^2 + dx^2 \cdot (u - x^1), \\
x^2 dx^2 &= dx^1 \cdot w + dx^2 \cdot v.
\end{aligned}$$

Proof. First of all we will prove that each commutation rule has a commutative optimal algebra.

Let a commutation rule is set by formulae I. It is enough to prove that the ideal I generated by the commutator $x^1 x^2 - x^2 x^1$ is consistent. We have

$$\begin{aligned}
A(x^1) &= \begin{pmatrix} u & w \\ v & \lambda v + x^1 \end{pmatrix}; \\
A(x^2) &= \begin{pmatrix} w + x^2 & \lambda w \\ \lambda v & \lambda^2 v - \lambda u + w + \lambda x^1 + x^2 \end{pmatrix} \quad (4.1)
\end{aligned}$$

Therefore $A(x^2) = \lambda A(x^1) + (w + x^2 - \lambda u)E$, where E is a unit matrix and evidently

$$A([x^1, x^2]) = [A(x^1), A(x^2)] = [A(x^1), \lambda A(x^1) + (w + x^2 - \lambda u)E] = 0$$

in the factor algebra $F < x^1, x^2 > / \{x^1 x^2 = x^2 x^1\}$, so $A_k^i I \subset I$. For partial derivatives we have

$$\begin{aligned}
D_1(x^1 x^2 - x^2 x^1) &= x^2 + A_1^k(x^1) D_k(x^2) - A_1^k(x^2) D_k(x^1) = \\
&= x^2 + A_1^2(x^1) - A_1^1(x^2) = x^2 + w - (w + x^2) = 0, \\
D_2(x^1 x^2 - x^2 x^1) &= A_2^k(x^1) D_k(x^2) - x^1 - A_2^k(x^2) D_k(x^1) = \\
&= A_2^2(x^1) - x^1 - A_2^1(x^2) = \lambda v + x^1 - x^1 - \lambda v = 0
\end{aligned}$$

and all the commutation rules from series (I) have commutative optimal algebra.

Let a commutation rule is set by formula (II). We have

$$\begin{aligned}
A(x^1) &= \begin{pmatrix} x^1 + \mu v + v_1 & \lambda v \\ v & x^1 + v_1 \end{pmatrix}, \\
A(x^2) &= \begin{pmatrix} x^2 + \lambda v & \lambda v_1 \\ v_1 & x^2 + \lambda v - \mu v_1 \end{pmatrix}, \quad (4.2) \\
A([x^1, x^2]) &= [A(x^1), A(x^2)] = 0.
\end{aligned}$$

Therefore $A_k^i I \subset I$. Moreover

$$\begin{aligned}
D_1(x^1 x^2 - x^2 x^1) &= x^2 + A_1^2(x^1) - A_1^1(x^2) = 0, \\
D_2(x^1 x^2 - x^2 x^1) &= A_2^2(x^1) - x^1 - A_2^1(x^2) = 0,
\end{aligned}$$

and all the commutation rules given by the formulae (II) have commutative optimal algebra.

Let a commutation rule is set by the formulae (III). We have

$$A(x^1) = \begin{pmatrix} u & 0 \\ 0 & x^1 \end{pmatrix}, \quad A(x^2) = \begin{pmatrix} x^2 & 0 \\ 0 & v \end{pmatrix}. \quad (4.3)$$

Because $A(x^1)A(x^2) = A(x^2)A(x^1)$ then $A_k^i I \subset I$. Moreover

$$D_1(x^1 x^2 - x^2 x^1) = x^2 + A_1^2(x^1) - A_1^1(x^2) = 0,$$

$$D_2(x^1 x^2 - x^2 x^1) = A_2^2(x^1) - x^1 - A_2^1(x^2) = 0,$$

and all the commutation rules given by formulae (III) have commutative optimal algebra.

In the case (IV) we have

$$A(x^1) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad A(x^2) = \begin{pmatrix} x^2 & w \\ u - x^1 & v \end{pmatrix}. \quad (4.4)$$

It is evident that $A_k^i I \subset I$. Moreover

$$D_1(x^1 x^2 - x^2 x^1) = x^2 + A_1^2(x^1) - A_1^1(x^2) = 0,$$

$$D_2(x^1 x^2 - x^2 x^1) = A_2^2(x^1) - x^1 - A_2^1(x^2) = 0,$$

and all of the commutation rules in the theorem have commutative optimal algebra.

Conversely, let a commutation rule $x^i dx^j = dx^k A_k^{ij}$, where $A_k^{ij} = A_k^j(x^i)$, has commutative optimal algebra S . In this case

$$0 = D_k(x^i x^j - x^j x^i) = \delta_k^i x^j + A_k^{ij} - \delta_k^j x^i - A_k^{ji}$$

As a consequence we obtain necessary conditions (and they hold for $n > 2$):

$$A_k^{ij} = A_k^{ji} \quad \text{if } k \neq i \text{ and } k \neq j, \quad (4.5)$$

$$A_j^{ij} = x^i + A_j^{ji} \quad \text{if } j \neq i \text{ and } k = j. \quad (4.6)$$

If $n = 2$ then these conditions are reduced to the following two

$$A_2^{12} = x^1 + A_2^{21}; \quad A_1^{21} = x^2 + A_1^{12} \quad (4.7)$$

Also we have $A(x^1 x^2 - x^2 x^1) = 0$ i.e. the matrices $A^i = A_k^{is}$ and $A^j = A_k^{js}$ commutes in the ring of matrices over S :

$$A_k^{is} A_m^{jk} = A_k^{js} A_m^{ik} \quad (4.8)$$

Let consider these equalities in detail. All A_k^{is} have degree one i.e. they are in the space V . So the relations (4.8) have degree two and therefore have to belong to the second homogeneous component I_2 of the ideal $I(A)$. Let S_2 be a factor algebra \hat{R}/I_2 . This is a commutative algebra and relations the (4.8) are valid on it. As S is regular, the space I_2 is generated by commutator $x^1x^2 - x^2x^1$ and the algebra $S_2 = F[x^1, x^2]$ is the algebra of polynomials in two variables.

If one of the matrices $A(x^1), A(x^2)$ is scalar then (if necessary by renaming variables) we can suppose that $A(x^1) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ and the relations (4.7) show that $A(x^2)_1^1 = x^2$, $A(x^2)_2^1 = u - x^1$ and the commutation rule belongs to the series (IV). If both matrices $A(x^1), A(x^2)$ are diagonal than relations (4.7) immediately imply $A(x^1) = \begin{pmatrix} u & 0 \\ 0 & x^1 \end{pmatrix}$, $A(x^2) = \begin{pmatrix} x^2 & 0 \\ 0 & v \end{pmatrix}$ and the commutation rule belongs to the series (III). So under the following consideration we can suppose that no one of the matrices $A(x^1), A(x^2)$ is scalar and one of them is not diagonal matrix.

In the algebra of 2×2 matrices over the field of rational functions $K = F(x^1, x^2)$ the dimension over the field F of a centralizer of any non scalar matrix is equal to 2. It means that the centralizer of the matrix $A(x^1)$ is generated (over K) by two matrices $A(x^1)$ and E . It implies that we have the relation $A(x^2) = g \cdot A(x^1) + f \cdot E$ with $f, g \in K$. It implies $A_1^{22} = gA_1^{12}$, $A_2^{21} = gA_2^{11}$ and therefore $A_2^{21} \cdot A_1^{12} = A_1^{22}A_2^{11}$. By definition all the coefficients in the matrices are linear combinations of the variables so either $A_2^{21} = \lambda A_2^{11}, A_1^{22} = \lambda A_1^{12}$ or $A_1^{12} = \lambda A_2^{11}, A_1^{22} = \lambda A_2^{21}$, where $\lambda \in F$ or $\lambda = \infty$. The last case $\lambda = \infty$ means respectively $A_2^{11} = A_1^{12} = 0$ or $A_2^{11} = A_2^{21} = 0$. These two cases reduce to the cases $\lambda = 0$ by changing the variables $x^1 \leftrightarrow x^2$.

If $A_2^{21} = \lambda A_2^{11}$, $A_1^{22} = \lambda A_1^{12} - 1$, then a denotation $A_1^{11} = u$, $A_1^{12} = w$ implies $A_2^{21} = \lambda v$, $A_1^{22} = \lambda w$ and therefore $f = x^2 + w - \lambda u$. So $A_2^{22} = \lambda A_2^{12} + f = \lambda x^1 + \lambda^2 v + x^2 + w - \lambda u$. It means that the matrices $A(x^1), A(x^2)$ have the form (I). If $A_1^{12} = \lambda A_2^{11}$, $A_1^{22} = \lambda A_2^{21}$ then it is convenient to denote $A_2^{11} = v$, $A_2^{21} = v_1$. In this case $g = \frac{v_1}{v}$ and $f = A_1^{21} - \frac{v_1}{v}A_1^{11} = A_2^{22} - \frac{v_1}{v}A_2^{12}$ or $v_1(A_1^{11} - A_2^{12}) = v(A_1^{21} - A_2^{22})$. All the factors in the last relation are linear combinations of the variables, therefore $A_1^{11} - A_2^{12} = \mu v$, $A_1^{21} - A_2^{22} = \mu v_1$ where $\mu \in F$ ($\mu = \infty$ as $v \neq 0$ or $v_1 \neq 0$ because one of the matrices A^1, A^2 is not diagonal). Now relations (4.7) have the form $A_2^{12} = x^1 + v_1$, $A_1^{21} = x^2 + \lambda v$ which implies $A_1^{11} = x^1 + v_1 + \mu v$, and $A_2^{22} = x^2 + \lambda v - \mu v_1$. This means that we are in the situation (II). The theorem is proved. Q.E.D.

NOTE: It is an open problem to determine the optimal algebra for the commutation rules described above. We do not claim that the optimal algebra is a polynomial algebra in two variables $S_2 = F \langle x^1, x^2 \rangle / \langle I_2 \rangle \equiv F[x^1, x^2]$.

5. Acknowledgments. The authors are greatly indebted to Zbigniew Oziewicz for his active interest in the publication of this paper and for many stimulating conversations.

References

- [1] Borowiec A., Kharchenko V. K. and Oziewicz Z., (1993), *On free differentials on associative algebras*, in *Non Associative Algebras and Its Applications*, series *Mathematics and Applications*, ed. S. González, Kluwer Academic Publishers, Dordrecht, 1994, 43-56 (hep-th/9312023)
Borowiec A., Kharchenko V. K., (1994), *Coordinate calculi on associative algebras*, in *Quantum Groups: Formalism and Applications*, Ed. J. Lukierski et al, PWN, Warsaw 1994, 231-242 (hep-th/9501051)
- [2] Dimakis A. and Müller-Hoissen F., (1992), *Quantum mechanics as non-commutative symplectic geometry*, J. Phys. A: Math. Gen. **25** 5625-5648
Dimakis A., Müller-Hoissen F. and Striker T., (1993), *Non-commutative differential calculus and lattice gauge theory*, J. Phys. A: Math. Gen. **26** 1927-1949
- [3] Fox Ralph H. (1953), *Free differential calculus. I. Derivation in the free group ring*, Annals of Mathematics **57** (3) 547- 560
Fox Ralph H. (1954), *Free differential calculus II.*, Annals of Mathematics **59** (2) 196-210
Fox Ralph H. (1960), *Free differential calculus V.*, Annals of Mathematics **71** (3) 407-446
- [4] Kharchenko V. K. and Borowiec A., (1995), *Algebraic approach to calculi with partial derivatives*, to be published in Siberian Advances in Mathematics **5** (2) 1-28
- [5] Pusz W., (1989), *Twisted canonical anticommutation relations*, Reports on Mathematical Physics **27** 349-360
- [6] Pusz W. and Woronowicz S. L., (1989), *Twisted second quantization*, Reports on Mathematical Physics **27** 231-257
- [7] Wess J. and Zumino B., (1990), *Covariant differential calculus on the quantum hyperplane*, Nuclear Physics **18B** 303-312,
Proceedings, Supplements. Volume in honor of R. Stora.
- [8] Woronowicz S. L., (1989), *Differential calculus on compact matrix pseudogroups (quantum groups)*, Comm. Math. Phys. **122** 125-170.